OPERATIONS RESEARCH

Linear Programming: Simplex Method

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Simplex Method

The *simplex* is an important term in mathematics that represents an object in *n*-dimensional space connecting *n* + 1 points. In one dimension, a simplex is a line segment connecting two points; in two dimensions, it is a triangle formed by joining three points; in three dimensions, it is a four sided pyramid having four corners.

The concept of simplex method is similar to the graphical method. In graphical method, extreme points of the feasible solution space are examined to search for optimal solution at one of them. For LP problems with several variables, we may not be able to graph the feasible region, but the optimal solution will still lie at an extreme point of the many-sided, multidimensional figure (called an *n*-dimensional polyhedron) that represents the feasible solution space.

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The simplex method examines the extreme points in a systematic manner, repeating the same set of steps of the algorithm until an optimal solution is reached. It is for this reason that it is also called the *iterative method*.

Since the number of extreme points (corners or vertices) of feasible solution space are finite, the method assures an improvement in the value of objective function as we move from one iteration (extreme point) to another and achieve optimal solution in a finite number of steps and also indicates when an unbounded solution is reached.

Standard Form of An LP Problem

- All the constraints should be expressed as equations by adding slack or surplus and/or artificial variables.
- The right-hand side of each constraint should be made non-negative; if it is not, this should be done by multiplying both sides of the resulting constraint by – 1.
- The objective function should be of the maximization type.

Standard form of the LP problem is expressed as:

Optimize (Max or Min) $Z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n + 0s_1 + 0s_2 + \ldots + 0s_m$

subject to the linear constraints

$$
\begin{array}{lllllll}
a_{11} x_1 & + & a_{12} x_2 & + \ \ldots \ + & a_{1n} x_n & + & s_1 = b_1 \\
a_{21} x_1 & + & a_{22} x_2 & + \ \ldots \ + & a_{2n} x_n & + & s_2 = b_2 \\
. & & . & & . & & \\
. & & . & & . & & \\
. & & . & & . & & \\
a_{m1} x_1 & + & a_{m2} x_2 & + \ \ldots \ + & a_{mn} x_n & + & s_m = b_m
\end{array}
$$

and $x_1, x_2, \ldots, x_n, s_1, s_2, \ldots, s_m \geq 0$

Optimize (Max or Min) $Z = \mathbf{cx} + \mathbf{0s}$

subject to the linear constraints

$$\mathbf{Ax} + \mathbf{s} = \mathbf{b}$$

and $\quad \mathbf{x}, \mathbf{s} \geq \mathbf{0}$

where $\mathbf{c} = (c_1, c_2, \ldots, c_n)$ is the row vector, $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$, $\mathbf{b} = (b_1, b_2, \ldots, b_m)^T$ and $\mathbf{s} = (s_1, s_2, \ldots, s_m)$ are column vectors,

and
$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}$$

is the $m \times n$ matrix of coefficients of variables $x_1, x_2, \ldots, x_n$ in the constraints.

Three types of additional variables, namely

- slack variables (s)
- surplus variables ($-s$), and
- artificial variables (A)

are added in the given LP problem to convert it into the standard form for the following reasons:

(a) These variables allow us to convert inequalities into equalities, thereby converting the given LP problem into a form that is amenable to algebraic solution.

(b) These variables permit us to make a more comprehensive economic interpretation of a final solution.

(c) Help us to get an initial feasible solution represented by the columns of the identity matrix.

Types of Constraint	*Extra Variable Needed*	*Coefficient of Extra Variables in the Objective Function*		*Presence of Extra Variables in the Initial Solution Mix*
		Max Z	*Min Z*	
Less than or equal to (≤)	A slack variable is added	0	0	Yes
Greater than or equal to (≥)	A surplus variable is subtracted, and	0	0	No
	an artificial variable is added	– *M*	+ *M*	Yes
Equal to (=)	Only an artificial variable is added.	– *M*	+ *M*	Yes

Simplex Algorithm (Maximization Case)

Step 1: Formulation of the mathematical model

a) Formulate the mathematical model of the given linear programming problem.

b) If the objective function is of minimization, then convert it into one of maximization by using the following relationship

$$\text{Minimize } Z = -\text{ Maximize } Z^*$$

where $Z^* = -Z$.

c) Check whether all the b_i ($i = 1, 2, \ldots, m$) values are positive. If any one of them is negative, then multiply the corresponding constraint by -1 in order to make $b_i > 0$. In doing so, remember to change a $\leq$ type constraint to a $\geq$ type constraint, and vice versa.

d) Express the mathematical model of the given LP problem in the standard form by adding additional variables to the left side of each constraint and assign a *z*ero-cost coefficient to these in the objective function.

e) Replace each *unrestricted* variable with the difference of two non-negative variables; replace each non-positive variable with a new non-negative variable whose value is the negative of the original variable.

Step 2*:* Set-up the initial solution

Write down the coefficients of all the variables in the LP model in the tabular form, as shown in Table, to get an initial basic feasible solution [$\mathbf{x}_B = \mathbf{B}^{-1}\,\mathbf{b}$].

Initial Simplex Table

		$C_j \rightarrow$	c_1	c_2	…	c_n	0	0	…	0
Coefficient of Basic Variables ($\boldsymbol{c_B}$)	*Variables in Basis* $\boldsymbol{B}$	*Value of Basic Variables* $\boldsymbol{b}$ (= $\boldsymbol{x_B}$)	*Variables* x_1	x_2	…	x_n	s_1	s_2	. . .	s_m
c_{B1}	s_1	$x_{B1} = b_1$	a_{11}	a_{12}	…	a_{1n}	1	0	…	0
c_{B2}	s_2	$x_{B2} = b_2$	a_{21}	a_{22}	…	a_{2n}	0	1	…	0
.	.	.	.	.		.	.	.	.	.
.	.	.	.	.		.	.	.	.	.
.	.	.	.	.		.	.	.	.	.
c_{Bm}	s_m	$x_{Bm} = b_m$	a_{m1}	a_{m2}	…	a_{mn}	0	0	0	0
$Z = \Sigma\, c_{Bi} x_{Bi}$ = (B.V. coefficients) × (Values of B.V.)	$Z_j = \Sigma\, c_{Bi} x_j$ = Σ (B.V.coefficients) × (*j*th column of data matrix)		0	0	…	0	0	0	…	0
	$c_j - z_j$		$c_1 - z_1$	$c_2 - z_2$	…	$c_n - z_n$	0	0	…	0

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The values z_j represent the amount by which the value of objective function Z would be decreased (or increased) if one unit of given variable is added to the new solution. Each of the values in the $c_j - z_j$ row represents the net amount of increase (or decrease) in the objective function that would occur when one unit of variable represented by the column head is introduced into the solution. That is:

$c_j - z_j$ (net effect) = c_j (incoming unit profit/cost) – z_j (outgoing total profit/cost)

where z_j = Coefficient of basic variables column × Exchange coefficient column j

Step 3: Test for optimality

- If all $c_j - z_j \leq 0$, then the basic feasible solution is optimal.
- If at least one column of the coefficients matrix (i.e. $\mathbf{a}_k$) for which $c_k - z_k > 0$ and all elements are negative (i.e. $a_{ik} < 0$), then there exists an unbounded solution to the given problem.
- If at least one $c_j - z_j > 0$ and each of these has at least one positive element (i.e. a_{ij}) for some row, then it indicates that an improvement in the value of objective function Z is possible.

Step 4: Select the variable to enter the basis

If Case (iii) of Step 3 holds, then select a variable that has the largest $c_j - z_j$ value to enter into the new solution. That is,

$$c_k - z_k = \text{Max } \{(c_j - z_j);\ \ c_j - z_j > 0\}$$

The column to be entered is called the *key* or *pivot* column. Obviously, such a variable indicates the largest per unit improvement in the current solution. Obviously, such a variable indicates the largest per unit improvement in the current solution.

Step 5: Test for feasibility (variable to leave the basis)

Each number in $\mathbf{x_B}$-column (i.e. b_i values) is divided by the corresponding (but positive) number in the key column and a row is selected for which this ratio, [(*constant column*)/(*key column*)] is non-negative and minimum. This ratio is called the *replacement* (*exchange*) *ratio.* That is,

$$\frac{x_{Br}}{a_{rj}} = \text{Min}\left\{\frac{x_{Bi}}{a_{rj}};\ a_{rj} > 0\right\}$$

This ratio limits the number of units of incoming variable that can be obtained from the exchange. *It may be noted here that division by negative or zero element in key column is not permitted.*

Step 6: Finding the new solution

- If the key element is 1, then the row remains the same in the new simplex table.
- If the key element is other than 1, then divide each element in the key row (including elements in $\mathbf{x_B}$-column) by the key element, to find the new values for that row.
- The new values of the elements in the remaining rows for the new simplex table can be obtained by performing elementary row operations on all rows so that all elements except the key element in the key column are *zero*.

In other words, for each row other than the key row, we use the formula:

$$\text{Number in new row} = \left(\begin{array}{c}\text{Number in}\\ \text{old row}\end{array}\right) \pm \left(\begin{array}{c}\text{Number above or below}\\ \text{Key element}\end{array}\right)\left(\begin{array}{l}\text{Corresponding number in the new row,}\\ \text{that is row replaced in Step 6 (ii)}\end{array}\right)$$

The new entries in $\mathbf{c_B}$ (coefficient of basic variables) and $\mathbf{x_B}$ (value of basic variables) columns are updated in the new simplex table of the current solution.

Step 7: Repeat the procedure

Go to Step 3 and repeat the procedure until all entries in the $c_j - z_j$ row are either negative or *z*ero.

Example 1: Use the simplex method to solve the following LP problem.

Maximize $Z = 3x_1 + 5x_2 + 4x_3$

subject to the constraints

$$2x_1 + 3x_2 \leq 8$$

$$2x_2 + 5x_3 \leq$$

10

$$3x_1 + x_2 + 4x_3 \leq 15$$

and $x_1, \quad x_2, \quad x_3 > 0$

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Solution Step 1: Introducing non-negative slack variables s_1, s_2 and s_3 to convert inequality constraints to equality. Then the LP problem becomes

Maximize $Z = 3x_1 + 5x_2 + 4x_3 + 0s_1 + 0s_2 + 0s_3$

subject to the constraints

$$
\begin{aligned}
2x_1 + 3x_2 \qquad\quad + s_1 &= 8 \\
2x_2 + 5x_3 + s_2 &= 10 \\
3x_1 + 2x_2 + 4x_3 + s_3 &= 15
\end{aligned}
$$

and $x_1,\ x_2,\ x_3,\ s_1,\ s_2,\ s_3 \geq 0$

Step 2: Since all b_i (RHS values) > 0, (i = 1, 2, 3) we can choose initial basic feasible solution as:

$$x_1 = x_2 = x_3 = 0;\ s_1 = 8,\ s_2 = 10,\ s_3 = 15 \quad \text{and Max } Z = 0$$

This solution can also be read from the initial simplex Table by equating row wise values in the basis (B) column and solution values ($\mathbf{x_B}$) column.

Step 3: To see whether the current solution given in Table 4.3 is optimal or not, calculate

$$c_j - z_j = c_j - \mathbf{c_B B^{-1} a}_j = c_j - \mathbf{c_B y}_j$$

for non-basic variables x_1, x_2 and x_3 as follows:

z_j = (Basic variable coefficients, $\mathbf{c_B}$) × (jth column of data matrix)

That is, $z_1 = 0\,(2) + 0\,(0) + 0\,(3) = 0$ for x_1-column

$z_2 = 0\,(3) + 0\,(2) + 0\,(2) = 0$ for x_2-column

$z_3 = 0\,(0) + 0\,(5) + 0\,(4) = 0$ for x_3-column

These z_j values are now subtracted from c_j values to calculate net profit from introducing one unit of each variable x_1, x_2 and x_3 into the new solution mix.

$$c_1 - z_1 = 3 - 0 = 3$$

$$c_2 - z_2 = 5 - 0 = 5$$

$$c_3 - z_3 = 4 - 0 = 4$$

The z_j and $c_j - z_j$ rows are added into the Initial Solution Table.

The values of basic variables, s_1, s_2 and s_3 are given in the *solution values* ($\mathbf{x_B}$) column of Table 4.3. The remaining variables which are non-basic at the current solution have *z*ero value. The value of objective function at the current solution is given by

$$Z = \text{(Basic variable coefficients, } \mathbf{c_B}) \times \text{(Basic variable values, } \mathbf{x_B})$$

$$= 0\,(8) + 0\,(10) + 0\,(15) = 0$$

Initial Solution Table

Profit per Unit c_b	Variables in Basis B	Solution Values B (= x_{b})	x_1	x_2	x_3	s_1	s_2	s_3	Min Exchange Ratio x_B/x_2
		$c_j \rightarrow$	3	5	4	0	0	0	
0	s_1	8	2	③	0	1	0	0	83 $\rightarrow$
0	s_2	10	0	2	5	0	1	0	10/2
0	s_3	15	3	2	4	0	0	1	15/2
Z = 0		z_j	0	0	0	0	0	0	
		$c_j - z_j$	3	5	4	0	0	0	
				$\uparrow$					

Since all $c_j – z_j$ ³ 0 (j = 1, 2, 3), the current solution is not optimal. Variable x_2 is chosen to enter into the basis as $c_2 – z_2 = 5$ is the largest positive number in the x_2-column, where all elements are positive. This means that for every unit of variable x_2, the objective function will increase in value by 5.The x_2-column is the key column.

Step 4: The variable to leave the basis is determined by dividing the values in the $\mathbf{x_B}$- column by the corresponding elements in the key column as shown in Initial Solution Table Since the exchange ratio, 8/3 is minimum in row 1, the basic variable s_1 is chosen to leave the solution (basis).

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Step 5: *(Iteration 1)* Since the key element enclosed in the circle in Table 4.3 is not 1, divide all elements of the key row by 3 to obtain new values of the elements in this row. The new values of the elements in the remaining rows for the new Table 4.4 are obtained by performing the following elementary row operations on all rows so that all elements except the key element 1 in the key column are zero.

$$R_1 \text{ (new)} \rightarrow R_1 \text{ (old)} \div 3 \text{ (}\textit{key element}\text{)}$$

$$\rightarrow (8/3, 2/3, 3/3, 0/3, 1/3, 0/3, 0/3) = (8/3, 2/3, 1, 0, 1/3, 0, 0)$$

... **Step 5:** *(Iteration 1)*

R_2 (new) $\rightarrow$ R_2 (old) – $2R_1$ (new)		R_3 (new) $\rightarrow$ $2R_3$ (old) – $2R_1$ (new)	
10 – 2 × 8/3 =	14/3	15 – 2 × 8/3 =	29/3
0 – 2 × 2/3 =	– 4/3	3 – 2 × 2/3 =	5/3
2 – 2 × 1 =	0	2 – 2 × 1 =	0
5 – 2 × 0 =	5	4 – 2 × 0 =	4
0 – 2 × 1/3 =	– 2/3	0 – 2 × 1/3 =	– 2/3
1 – 2 × 0 =	1	0 – 2 × 0 =	0
0 – 2 × 0 =	0	1 – 2 × 0 =	1

Improved Solution Table

		$c_j \rightarrow$	3	5	4	0	0	0	
Profit per Unit **C_B**	Variables in Basis **B**	Solution Values **b (= x_B)**	x_1	x_2	x_3	s_1	s_2	s_3	Min Ratio **x_B/** x_3
5	x_2	8/3	2/3	1	0	1/3	0	0	-
0	s_2	14/3	- 4/3	0	⑤	- 2/3	1	0	(14/3)/5 $\rightarrow$
0	s_2	29/3	5/3	0	4	- 2/3	0	1	(29/3)/4
Z = 40/3		z_j	10/3	5	0	5/3	0	0	
		$c_j - z_j$	- 1/3	0	4	- 5/3	0	0	
					$\uparrow$				

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An improved basic feasible solution can be read from Improved Solution Table as: x_2 = 8/3, s_2 = 14/3, s_3 = 29/3 and $x_1 = x_3 = s_1 = 0$. The improved value of the objective function is

$$\boldsymbol{Z} = (\text{Basic variable coefficients, } \mathbf{c_B}) \times (\text{Basic variable values, } \mathbf{x_B})$$

$$= 5\,(8/3) + 0\,(14/3) + 0\,(29/3) = 40/3$$

Once again, calculate values of $c_j - z_j$ in the same manner as discussed earlier to see whether the solution shown in Table 4.4 is optimal or not. Since $c_3 - z_3 > 0$, the current solution is not optimal.

Step 6: (*Iteration 2*) Repeat Steps 3 to 5. Table 4.5 is obtained by performing following row operations to enter variable x_3 into the basis and to drive out s_2 from the basis.

$$R_2\,(\text{new}) = R_2\,(\text{old}) \div 5\ (\textit{key element})$$

$$= (14/15,\ -4/15,\ 0,\ 1,\ -2/15,\ 1/5,\ 0)$$

R_2 (new) $\rightarrow$ R_3 (old) – $4R_2$ (new)
$29/3 - 4 \times 14/15 = 89/15$
$5/3 - 4 \times -4/15 = 41/15$
$0 - 4 \times 0 = 0$
$4 - 4 \times 0 = 0$
$-2/3 - 4 \times -2/15 = -2/15$
$0 - 4 \times 1/5 = -4/5$
$1 - 4 \times 0 = 0$

Improved Solution Table is completed by calculating the new z_j and $c_j - z_j$ values and the new value of objective function:

$$z_1 = 5\,(2/3) + 4\,(-4/15) + 0\,(41/15) = 34/15$$

$$z_4 = 5\,(1/3) + 4\,(-2/15) + 0\,(2/15) = 17/15$$

$$z_5 = 5\,(0) + 4\,(1/5) + 0\,(-4/5) = 4/5$$

$$c_1 - z_1 = 3 - 34/15 = 11/15 \quad \text{for } x_1\text{-column}$$

$$c_4 - z_4 = 0 - 17/15 = -17/15 \quad \text{for } s_1\text{-column}$$

$$c_5 - z_5 = 0 - 4/5 = -4/5 \quad \text{for } s_2\text{-column}$$

The new objective function value is given by

Z = (Basic variable coefficients, $\mathbf{c_B}$) × (Basic variable values, $\mathbf{x_B}$)

= 5 (8/3) + 4 (14/15) + 0 (89/15) = 256/15

The improved basic feasible solution is shown in Improved Solution Table

Improved Solution Table

Profit per Unit C_B	Variables in Basis **B**	Solution Values **b** ($= x_B$)	x_1	x_2	x_3	s_1	s_2	s_3	Min *Ratio* x_B / x_1
		$c_j \rightarrow$	3	5	4	0	0	0	
5	x_2	8/3	2/3	1	0	1/3	0	0	(8/3)/(2/3)
4	x_3	14/15	- 4/15	0	1	- 2/15	1/5	0	-
0	s_3	89/15	(41/15)	0	0	2/15	- 4/5	1	(89/15)/(41/15) →
Z = 256/15		z_j	34/15	5	4	17/15	4/5	0	
		$c_j = z_j$	11/15	0	0	- 17/15	- 4/5	0	
			↑						

Iteration 3 In Improved Solution Table since, $c_1 - z_1$ is still a positive value, the current solution is not optimal. Thus, the variable x_1 enters the basis and s_3 leaves the basis. To get another improved solution as shown in Table 4.5 performing following row operations in the same manner as discussed earlier.

R_3 (new) $\rightarrow R_3$ (old) × 15/41 (*key element*)

$\rightarrow$ (89/15 × 15/41, 41/15 × 15/41, 0 × 15/41, 0 × 15/41, – 2/15 × 15/41, – 4/5 × 15/41, 1 × 15/41)

$\rightarrow$ (89/41, 1, 0, 0, – 2/41, – 12/41, 15/41)

R_1 (new) → R_1 (old) – (2/3) R_3 (new)	R_2 (new) → R_2 (old) + (4/15) R_3 (new)
8/3 – 2/3 × 89/3 = 50/41	14/15 + 4/15 × 89/41 = 62/41
2/3 – 2/3 × 1 = 0	– 4/15 + 4/15 × 1 = 0
1 – 2/3 × 0 = 1	0 + 4/15 × 0 = 0
0 – 2/3 × 0 = 0	1 + 4/15 × 0 = 1
1/3 – 2/3 × – 2/41 = 15/41	– 2/15 + 4/15 × – 2/41 = – 6/41
0 – 2/3 × – 12/41 = 8/41	1/5 + 4/15 × – 12/41 = 5/41
0 – 2/3 × 15/41 = – 10/41	0 + 4/15 × 15/41 = 4/41

Optimal Solution Table

		$c_j \rightarrow$	3	5	4	0	0	0
Profit per Unit **C_B**	Variables in Basis **B**	Solution Values **b (= x_B)**	x_1	x_2	x_3	s_1	s_2	s_3
5	x_2	50/41	0	1	0	15/41	8/41	- 10/41
4	x_3	62/41	0	0	**1**	- 6/41	5/41	4/41
3	x_1	89/41	1	0	0	- 2/41	- 12/41	15/41
Z = 765/41		z_j	3	5	4	45/41	24/41	11/41
		$c_j - z_j$	0	0	0	- 45/41	- 24/41	- 11/41

In Optimal Solution Table all $c_j - z_j < 0$ for non-basic variables. Therefore, the optimal solution is reached with, $x_1 = 89/41$, $x_2 = 50/41$, $x_3 = 62/41$ and the optimal value of $Z = 765/41$.

Simplex Algorithm (Minimization Case)

In certain cases, it is difficult to obtain an initial basic feasible solution. Such cases arise

(i) when the constraints are of the $\leq$ type

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad , \qquad x_j \geq 0$$

but some right-hand side constants are negative [i.e. $b_i < 0$]. In this case after adding the non-negative slack variable s_i ($i = 1, 2, . . ., m$), the initial solution so obtained will be $s_i = - b_i$ for some i. It is not the feasible solution because it violates the non-negativity conditions of slack variables (i.e. $s_i \geq 0$).

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(ii) when the constraints are of the $\geq$ type

$$\sum_{j=1}^{n} a_{ij}\, x_j \geq b_i, \quad x_j \geq 0$$

In this case to convert the inequalities into equation form, adding surplus (negative slack) variables,

$$\sum_{j=1}^{n} a_{ij}\, x_j - s_i = b_i, \quad x_j \geq 0,\ s_i \geq 0$$

. . . Simplex Algorithm (Minimization Case)

Letting $x_j = 0$ ($j = 1, 2, . . ., n$), we get an initial solution $-s_i = b_i$ or $s_i = -b_i$. It is also not a feasible solution as it violates the non-negativity conditions of surplus variables (i.e. s_i ³ 0). In this case, we add artificial variables, A_i ($i = 1, 2, . . ., m$) to get an initial basic feasible solution. The resulting system of equations then becomes:

$$\sum_{j=1}^{n} a_{ij} x_j - s_i + A_i = b_i$$

$$x_j, s_i, A_i \geq 0, \quad i = 1, 2, . . ., m$$

. . . Simplex Algorithm (Minimization Case)

and has m equations and ($n + m + m$) variables (i.e. n decision variables, m artificial variables and m surplus variables). An initial basic feasible solution of the new system can be obtained by equating ($n + 2m – m$) = ($n + m$) variables equal to zero. Thus the new solution to the give LP problem is $A_i = b_i$ ($i = 1, 2, . . . , m$), which does not constitute a solution to the original system of equations because the two systems of equations are not equivalent. Thus to get back to the original problem, artificial variables must be dropped out of the optimal solution. There are two methods for eliminating these variables from the solution.

The Big-M Method

Assign a large undesirable (unacceptable penalty) coefficients to artificial variables from the objective function point of view. If objective function Z is to be minimized, then a very large positive price (called *penalty*) is assigned to each artificial variable. Similarly, if Z is to be maximized, then a very large negative price (also called *penalty*) is assigned to each of these variables. The penalty will be designated by $-M$ for a maximization problem and $+M$ for a minimization problem, where $M > 0$.

Steps of the algorithm

Step 1: Express the LP problem in the standard form by adding slack variables, surplus variables and artificial variables. Assign a *z*ero coefficient to both slack and surplus variables and a very large positive coefficient + *M* (minimization case) and – *M* (maximization case) to artificial variable in the objective function.

Step 2: The initial basic feasible solution is obtained by assigning zero value to original variables.

. . . Steps of the algorithm

Step 3: Calculate the values of $c_j - z_j$ in last row of the simplex table and examine these values.

- If all $c_j - z_j$ ³ 0, then the current basic feasible solution is optimal.
- If for a column, k, $c_k - z_k$ is most negative and all entries in this column are negative, then the problem has an unbounded optimal solution.
- If one or more $c_j - z_j < 0$ (minimization case), then select the variable to enter into the basis with the largest negative $c_j - z_j$ value (largest per unit reduction in the objective function value). This value also represents opportunity cost of not having one unit of the variable in the solution. That is,

$$c_k - z_k = \text{Min}\,\{c_j - z_j : c_j - z_j < 0\}$$

. . . Steps of the algorithm

Step 4: Determine the key row and key element in the same manner as discussed in the simplex algorithm of the maximization case.

Step 5: Continue with the procedure to update solution at each iteration till optimal solution is obtained.

. . . Steps of the algorithm

At any iteration of the simplex algorithm any one of the following cases may arise:

- If at least one artificial variable is present in the basis with zero value and the coefficient of M in each $c_j - z_j$ (j = 1, 2, . . ., n) values is non-negative, then the given LP problem has no solution. That is, the current basic feasible solution is degenerate.
- If at least one artificial variable is present in the basis with positive value and the coefficient of M in each $c_j - z_j$ (j = 1, 2, . . ., n) values is non-negative, then given LP problem has no optimum basic feasible solution. In this case, the given LP problem has a *pseudo optimum* basic feasible solution

Example: Use the penalty (Big-*M*) method to solve the following LP problem.

Minimize $Z = 5x_1 + 3x_2$

subject to the constraints

$$2x_1 + 4x_2 \leq 12$$
$$2x_1 + 2x_2 = 10$$
$$5x_1 + 2x_2 \geq 10$$

and $x_1, x_2 \geq 0.$

Solution: Introducing slack variable s_1, surplus variable s_2 and artificial variables A_1 and A_2 in the constraints of the given LP problem. The standard form of the LP problem is stated as follows:

Minimize $Z = 5x_1 + 3x_2 + 0s_1 + 0s_2 + MA_1 + MA_2$

subject to the constraints

$$2x_1 + 4x_2 + s_1 = 12$$
$$2x_1 + 2x_2 + A_1 = 10$$
$$5x_1 + 2x_2 - s_2 + A_2 = 10$$

And $x_1, x_2, s_1, s_2, A_1, A_2 \geq 0$

. . . The Big-M Method

An initial basic feasible solution is obtained by letting $x_1 = x_2 = s_2 = 0$. Therefore, the initial basic feasible solution is: $s_1 = 12$, $A_1 = 10$, $A_2 = 10$ and Min $Z = 10M + 10M = 20M$. Here it may be noted that the columns which corresponds to current basic variables and form the basis (identity matrix) are s_1 (slack variable), A_1 and A_2 (both artificial variables). The initial basic feasible solution is given in Initial solution Table.

Since the value $c_1 - z_1 = 5 - 7M$ is the smallest value, therefore x_1 becomes the entering variable. To decide which basic variable should leave the basis, the minimum ratio is calculated as shown in Initial solution Table.

Initial solution Table

		$c_j \rightarrow$	5	3	0	0	M	M	
Cost per Unit C_B	Variables in Basis **B**	Solution Values **b (=** x_B**)**	x_1	x_2	s_1	s_2	A_1	A_2	Min *Ratio* x_B / x_1
0	s_1	12	2	4	1	0	0	0	12/2 = 6
M	A_1	10	2	2	0	0	1	0	10/2 = 6
M	A_2	10	⑤	2	0	- 1	0	1	10/5 = 2 →
$Z = 20M$		z_j	$7M$	$4M$	0	$-M$	M	M	
		$c_j - z_j$	$5 - 7M$	$3 - 4M$	0	M	0	0	
			↑						

. . . The Big-M Method

Iteration 1: Introduce variable x_1 into the basis and remove A_2 from the basis by applying the following row operations. The new solution is shown in Improved Solution Table.

R_3 (new) $\rightarrow$ R_3 (old) $\div$ 5 (*key element*); R_2 (new) $\rightarrow$ R_2 (old) – $2R_3$ (new).

R_1 (new) $\rightarrow$ R_1 (old) – $2R_3$ (new).

Improved Solution Table

		$c_j \rightarrow$	5	3	0	0	*M*	
Cost per Unit C_B	Variables in Basis **B**	Solution Values **b** $(= x_B)$	x_1	x_2	s_1	s_2	A_1	Min *Ratio* x_B / x_2
0	s_1	8	0	16/5	1	2/5	0	8/(16/5) = 5/2 →
M	A_1	6	0	6/5	0	2/5	1	6/(6/5) = 5
5	x_1	2	1	2/5	0	- 1/5	0	2/(2/5) = 5 →
Z = 10 + 6*M*		z_j	5	(6M/5) + 2	0	(2*M*/5) – 1	*M*	
		c_j - z_j	0	(- 6M/5) + 1	0	(- 2*M*/5) + 1	0	
				↑				

. . . The Big-M Method

Iteration 2: Since the value of $c_2 - z_2$ in Table 4.23 is largest negative value, variable x_2 is chosen to enter into the basis. For introducing variable x_2 into the basis and to remove s_1 from the basis we apply the following row operations. The new solution is shown in Improved Solution Table.

R_1 (new) $\rightarrow$ R_1 (old) × 5/16 (*key element*); R_2 (new) $\rightarrow$ R_2 (old) – (6/5) R_1 (new).

R_3 (new) $\rightarrow$ R_3 (old) – 2/5 R_1 (new).

Improved Solution Table

		$c_j \rightarrow$	5	3	0	0	M	
Cost per Unit C_B	**Variables** in Basis **B**	Solution Values **b (= x_B)**	x_1	x_2	s_1	s_2	A_1	Min *Ratio* x_B / s_2
3	x_2	5/2	0	1	5/16	1/8	0	(5/2)/(1/8) = 40
M	A_1	3	0	0	- 3/8	1/4	1	3/(1/4) = 12 →
5	x_1	1	1	0	- 1/8	- 1/4	0	-
$Z = 25/2 + 3M$		z_j	5	3	$- 3M/8 + 5/16$	$M/4 - 7/8$	M	
		$c_j - z_j$	0	0	$- 3M/8 - 5/16$	$- M/4 + 7/8$	0	
						↑		

. . . The Big-M Method

Iteration 3: As $c_4 - z_4 < 0$ in s_2-column, current solution is not optimal. Thus, introduce s_2 into the basis and remove A_1 from the basis by apply the following row operations:

R_2 (new) $\rightarrow$ R_2 (old) × 4 (*key element*); R_1 (new) $\rightarrow$ R_1 (old) – (1/8) R_2 (new)

R_3 (new) $\rightarrow$ R_3 (old) + (1/4) R_2 (new).

The new solution is shown in Optimal Solution Table.

Optimal Solution Table

		$c_j \rightarrow$	5	3	0	0
Cost per Unit **C_B**	Variables in Basis **B**	Solution Values **b (= x_B)**	x_1	x_2	s_1	s_2
3	x_2	1	0	1	1/2	0
0	s_2	12	0	0	- 3/2	1
5	x_1	4	1	0	- 1/2	0
Z = 23		z_j	5	3	- 1	0
		$c_j - z_j$	0	0	1	0

In above table, all $c_j - z_j \geq 0$. Thus an optimal solution is arrived at with value of variables as: $x_1 = 4$, $x_2 = 1$, $s_1 = 0$, $s_2 = 12$ and Min $Z = 23$.

. . . The Big-M Method

Example: Use penalty (Big-M) method to solve the following LP problem.

Maximize $Z = x_1 + 2x_2 + 3x_3 - x_4$

subject to the constraints

$$x_1 + 2x_2 + 3x_3 = 15$$

$$2x_1 + x_2 + 5x_3 = 20$$

$$x_1 + 2x_2 + x_3 + x_4 = 10$$

and $x_1, x_2, x_3, x_4 \geq 0$

. . . The Big-M Method

Solution: Since all constraints of the given LP problem are equations, therefore adding only artificial variables A_1 and A_2 in the constraints. The standard form of the problem is then stated as follows:

Maximize $Z = x_1 + 2x_2 + 3x_3 - x_4 - MA_1 - MA_2$

subject to the constraints

$$x_1 + 2x_2 + 3x_3 + A_1 = 15$$

$$2x_1 + x_2 + 5x_3 + A_2 = 20$$

$$x_1 + 2x_2 + x_3 + x_4 = 10$$

and $x_1, x_2, x_3, x_4, A_1, A_2 \geq 0$

An initial basic feasible solution is given in Initial Solution Table.

Initial Solution Table

		$c_j \rightarrow$	1	2	3	- 1	$-M$	$-M$	
Profit per Unit **C_B**	Variables in Basis **B**	Solution Values **b (= x_B)**	x_1	x_2	x_3	x_4	A_1	A_2	Min *Ratio* **x_B / x_3**
$-M$	A_1	15	1	2	3	0	1	0	$15/3 = 5$
$-M$	A_2	20	2	1	⑤	0	0	1	$20/5 = 4 \rightarrow$
-1	x_4	10	1	2	1	1	0	0	$10/1 = 10$
$Z = -35M - 10$	z_j	$-3M - 1$	$-3M - 2$	$-8M - 1$	-1	$-M$	$-M$		
		$c_j - z_j$	$3M + 2$	$3M + 4$	$8M + 4$	0	0	0	
					↑				

. . . The Big-M Method

Since the value of $c_3 - z_3$ in Table 4.25 is largest positive, the variable x_3 is chosen to enter into the basis. To get an improved basic feasible solution, apply the following row operations for entering variable x_3 into the basis and removing variable A_2 from the basis.

R_2 (new) $\rightarrow$ R_1 (old) $\div$ 5 (*key element*) ; R_1 (new) $\rightarrow$ R_1 (old) – 3 R_2 (new).

R_3 (new) $\rightarrow$ R_3 (old) – R_2 (new)

The new solution is shown in Improved Solution Table.

Improved Solution Table

Profit per Unit C_B	Variables in Basis **B**	Solution Values **b (= x_B)**	x_1	x_2	x_3	x_4	A_1	Min Ratio x_B / x_2
	$c_j \rightarrow$		1	2	3	–1	–M	
$-M$	A_1	3	– 1/51	7/5	0	0	1	$\frac{3}{7/5} = \frac{15}{7}$ →
3	x_3	4	2/5	1/5	1	0	0	$\frac{4}{1/5} = 20$
– 1	x_4	6	3/5	9/5	0	1	0	$\frac{6}{9/5} = \frac{30}{9}$
$Z = -3M + 6$		z_j	$M/5 + 3/5$	$-7M/5 - 6/5$	3	–1	$-M$	
		$c_j - z_j$	$-M/5 - 2/5$	$7M/5 + 16/5$ ↑	0	0	0	

. . . The Big-M Method

The solution shown in the Table, is not optimal because $c_2 - z_2$ is positive. Thus, applying the following row operations for entering variable x_2 into the basis and removing variable A_1 from the basis,

R_1 (new) $\rightarrow$ R_1 (old) × (5/7) (*key element*); R_2 (new) $\rightarrow$ R_2 (old) – (1/5) R_1 (new).

R_3 (new) $\rightarrow$ R_3 (old) – (9/5) R_1 (new).

The new solution is shown in Improved Solution Table.

Improved Solution Table

		$c_j \rightarrow$	1	2	3	– 1	
Profit per Unit **C_B**	Variables in Basis **B**	Solution Values **b (= x_B)**	x_1	x_2	x_3	x_4	Min *Ratio* **x_B/ x_1**
2	x_2	15/7	– 1/7	1	0	0	—
3	x_3	25/7	3/7	0	1	0	25/7 X 7/3 = 25/3
– 1	x_4	15/7	(6/7)	0	0	1	15/7 X 7/6 = 15/6 →
Z = 90/7		z_j	1/7	2	3	– 1	
		$c_j - z_j$	6/7 ↑	0	0	0	

. . . The Big-M Method

Once again, the solution shown in theTable, is not optimal as $c_1 - z_1 > 0$ in x_1–column. Thus, applying the following row operations for entering variable x_1 into the basis and removing variable x_4 from the basis,

R_3 (new) $\rightarrow$ R_3 (old) × (7/6) (*key element*) ; R_1 (new) $\rightarrow$ R_1 (old) + (1/7) R_3 (new).

R_2 (new) $\rightarrow$ R_2 (old) – (3/7) R_3 (new).

The new solution is shown in optimal solution Table.

Optimal Solution Table

Profit per Unit C_B	Variables in Basis B	$c_j \rightarrow$ Solution Values $b\ (= x_B)$	1 x_1	2 x_2	3 x_3	– 1 x_4
2	x_2	15/6	0	1	0	1/6
3	x_3	15/6	0	0	1	– 3/6
1	x_1	15/6	1	0	0	7/6
Z = 15		z_j	1	2	3	0
		$c_j - z_j$	0	0	0	– 1

In Optimal Solution Table, all $c_j - z_j \leq 0$. Thus, an optimal solution has been arrived at with values of variables as: x_1 = 15/6, x_2 = 15/6, x_3 = 15/6 and Max Z = 15.